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Asymptotic theory for the almost-highest solitary wave

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The behaviour of the energy in a steep solitary wave as a function of the wave height has a direct bearing on the breaking of solitary waves on a gently shoaling beach. Here it is shown that the speed, energy and momentum of a steep solitary wave in water of finite depth all behave in an oscillatory manner as functions of the wave height and as the limiting height is approached. Asymptotic formulae for these and other wave parameters are derived by means of a theory for the 'almost-highest wave' similar to that formulated previously for periodic waves in deep water (Longuet-Higgins & Fox 1977, 1978). It is demonstrated that the theory fits very precisely some recent calculations of solitary waves by Tanaka (1995).

1. Introduction

The study of steep solitary waves in a homogeneous fluid of uniform depth has had an interesting history, stimulated in part by its application to the breaking of ocean waves in shallow water. Early calculations of the highest solitary wave, including those by Yamada (1957), Lenau (1966) and Witting (1975), have been summarized in a review paper by Miles (1980, §4). Probably the most accurate calculation was later carried out by Williams (1981) whose numerical results have been confirmed to at least five places of decimals by Evans & Dörr (1991).

The most interesting theoretical results, however, concern the behaviour of a solitary wave at wave heights slightly less than the maximum. For many years it was generally assumed that the speed and energy of a solitary wave in a given depth of water would increase monotonically with the wave height. However, in 1974 it was discovered that many integral properties of solitary waves, including their speed and energy, attain maxima at wave heights less than that of the limiting wave (Longuet-Higgins & Fenton 1974). These authors used a series expansion with convergence accelerated by Padé approximants. Their conclusion was later confirmed by an independent calculation using an integral equation (Byatt-Smith & Longuet-Higgins 1976). The latter also gave a physical explanation for the phenomenon. Meanwhile similar results were noted for periodic irrotational waves in water of infinite depth (Longuet-Higgins 1975).

All the above calculations experienced difficulties for waves close to the limiting steepness. A significant advance came with the introduction of a theory for the 'almost-highest' wave by Longuet-Higgins & Fox (1977, 1978, referred to herein as LHF1 and LHF2). In LHF1 it was shown that any steady irrotational wave having a small radius of curvature R (not zero) at the crest would possess an asymptotic form of flow near the crest whose lengthscale was proportional to R. Moreover, this limiting 'inner solution' contained oscillatory terms which became small at distances from the crest that were large compared to R. This accounted for the fact that surface slopes slightly greater than 30° had been found in numerical calculations of steep, steady waves. Moreover, when the 'inner solution' was matched asymptotically to the rest of a periodic wave, as in LHF2, explicit formulae were found for the integral properties which displayed an oscillatory behaviour. These showed that the wave energy, for example, should have not only a single maximum but an infinite sequence of stationary values – alternating maxima and minima – at wave heights close to the limiting height.

In LHF2 the calculations were given in detail only for periodic waves in deep water, although it is evident from that paper that similar calculations were possible in principle for other types of Stokes waves, including waves of solitary type. In fact such calculations were already contained in the PhD thesis by Fox (1977). These, however, were not published, and meanwhile the existence of a second stationary value (or first minimum) for each of the integral quantities was rediscovered by Evans & Dörr (1991) using a different integral equation to that derived by previous authors.

Interest in these results was further enhanced by the demonstration by Tanaka (1986) that a solitary wave first becomes unstable at the critical value of the wave height corresponding to the first energy maximum. This result was to be anticipated from a corresponding conclusion for periodic waves (Tanaka 1983; Saffman 1985). The nonlinear development of this instability has been shown to lead directly to wave breaking, if the initial distance is of one particular sign; if it is of the opposite sign the wave undergoes a transition to a lower (stable) solitary wave having almost the same energy†. To conserve both energy and momentum a small solitary wave is shed into the 'tail' (see Tanaka *et al.* 1987).

Recently Tanaka (1995) has investigated the possibility of a second mode of instability occurring at the second stationary value (or first minimum) of the energy in a solitary wave. Details of these calculations are given by Longuet-Higgins & Tanaka (1996). To confirm the calculations we propose in the present paper to provide an account of the asymptotic theory for the almost-highest wave which was given in Fox (1977) but not previously published. The calculations are slightly modified so as to take account of the more recent results of Williams (1981).

The plan of the paper is as follows. Basically the approach is to work downwards from the highest wave, as in LHF1 and LHF2. In §2, therefore, we first present a calculation of the highest solitary wave. Then in §3 we describe the general method for matching the flow at the crest of an almost-highest wave to the flow in the rest of the wave. This involves making a first correction to the outer flow, which is done in §4, resulting in the asymptotic expressions for the phase speed, energy and momentum which are stated in §5. The constants in these asymptotic expressions are, however, valid only to one significant figure. In §6, an examination of Tanaka's (1995) data shows that the latter conform very closely to the asymptotic forms found in §5. By fitting the formulae to the data, the appropriate constants are obtained to at least two significant figures. The resulting expressions give the second and third stationary values of the energy and other wave properties to a high degree of accuracy.

2. The highest solitary wave

In this Section we carry out the initial step, namely the calculation of the highest solitary wave, having a 120° angle at the crest. We here use a method similar to that

† A somewhat similar interpretation of the observed intermittency in spilling breakers was given by Longuet-Higgins (1976).

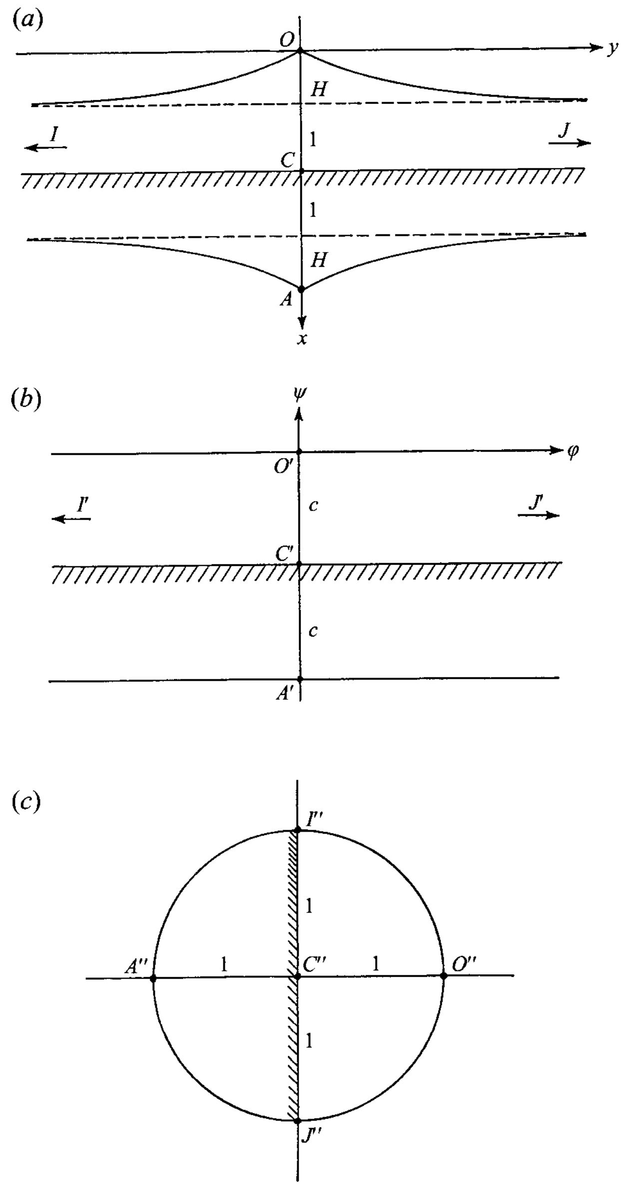

FIGURE 1. The highest solitary wave: (*a*) physical plane, (*b*) plane of $\chi = \phi + i\psi$, (*c*) plane of ζ.

of Lenau (1966), which will introduce the basic transformations needed later for the calculation of the almost-highest wave.

Let us choose units of mass, length and time so that the density ρ, gravitational acceleration g and undisturbed depth of water D are all equal to unity. The wave travels to the left with speed c. We take axes moving with the wave speed, with the x-axis vertically downwards and the y-axis horizontally to the right (see figure 1 *a*), and write z for the complex space coordinate $(x+iy)$. It is convenient to continue the flow analytically across the bottom to give a reflection of the physical wave. The condition to be satisfied on the reflection of the free surface will then be similar to that on the original free surface, but with the direction of gravity reversed.

Writing $\chi = \phi + \mathrm{i}\psi$ for the complex velocity potential, we may map the flow region

$$-\infty < \phi < \infty, \quad -2c < \psi < 0 \tag{2.1}$$

(see figure 1*b*) onto a circle in the ζ-plane by the transformation

$$\zeta = -\mathrm{i}\frac{\exp(\pi\chi/2c)+\mathrm{i}}{\exp(\pi\chi/2c)-\mathrm{i}} \tag{2.2}$$

or

$$\chi = \frac{2c}{\pi}\ln\left[\mathrm{i}\frac{\zeta-\mathrm{i}}{\zeta+\mathrm{i}}\right], \tag{2.3}$$

see figure 1(*c*). The bottom $x = 1+H$, $\psi = -c$ corresponds to the diameter $\mathrm{Re}(\zeta) = 0$ of the circle, while the original and reflected free surfaces correspond to the right and left halves of the circumference respectively. From (2.3) we have the asymptotic relations

$$\left.\begin{aligned} \chi &= \frac{2c\mathrm{i}}{\pi}(\zeta-1)+O(\zeta-1)^2, \quad \zeta\to+1, \\ \chi &= -2c\mathrm{i}\left[1-\frac{1}{\pi}(\zeta+1)\right]+O(\zeta+1)^2, \quad \zeta\to-1 \end{aligned}\right\} \tag{2.4}$$

near the crest and its reflection respectively, and

$$\left.\begin{aligned} \exp(\pi\chi/2c) &\sim \tfrac{1}{2}(\zeta-\mathrm{i}), \quad \zeta\to\mathrm{i}, \\ \exp(-\pi\chi/2c) &\sim \tfrac{1}{2}(\zeta+\mathrm{i}), \quad \zeta\to-\mathrm{i} \end{aligned}\right\} \tag{2.5}$$

far to the left and right of the crest.

At infinity the flow tends to the uniform stream:

$$z \sim \frac{\mathrm{i}\chi}{c}+\text{constant}, \quad \phi\to\pm\infty, \tag{2.6}$$

while near the crest and its reflection we have the Stokes corner flows

$$\left.\begin{aligned} z &\sim (\tfrac{3}{2}\mathrm{i}\chi)^{2/3}, \quad \chi\to0, \\ z &\sim 2(H+1)-(3c-\tfrac{3}{2}\mathrm{i}\chi)^{2/3}, \quad \chi\to-2\mathrm{i}c. \end{aligned}\right\} \tag{2.7}$$

Hence from equations (2.4) we see that

$$\frac{z-\mathrm{i}\chi/c+H(\zeta-1)}{(1-\zeta^2)^{2/3}} \tag{2.8}$$

must be finite on the free surface and analytic when $|\zeta| < 1$. So this function will have a power series expansion about $\zeta = 0$ which we assume to converge absolutely and uniformly when $|\zeta| \leqslant 1$.

However, in practice the convergence is rather slow in this case, and it is advisable to include explicitly the next terms in the asymptotic forms as $\phi\to\pm\infty$ or $\chi\to0$ or $-2\mathrm{i}c$. Thus from LHF1 we have

$$z \sim (\tfrac{3}{2}\mathrm{i}\chi)^{2/3}+B(\mathrm{i}\chi)^{-1/3-\lambda}, \tag{2.9}$$

where B is a constant and λ is the largest root of the transcendental equation

$$\frac{\pi\lambda}{2}\tan\frac{\pi\lambda}{2} = \frac{\pi}{2\sqrt{3}} \tag{2.10}$$

such that $\lambda < -1$, that is to say $\lambda = -1.8027$. So from equations (2.4) we take

$$z \sim \left[\frac{3c}{\pi}(1-\zeta)\right]^{2/3} + B\left[\frac{2c}{\pi}(1-\zeta)\right]^{-1/3-\lambda} \tag{2.11}$$

and a similar expression for z as $\zeta \to -1$. On the other hand as $\phi \to \pm\infty$ we know from Lamb (1932, §250) that

$$z \sim \frac{\mathrm{i}\chi}{c} + \text{constant} + \mathrm{i}C\exp\left[\mp m(\chi/c+\mathrm{i})\right], \tag{2.12}$$

where C is a constant and m is the smallest positive root of

$$m^{-1}\tan m = c^2. \tag{2.13}$$

So from equations (2.5) we have

$$z \sim \frac{\mathrm{i}\chi}{c} + \text{constant} + \mathrm{i}C'(\zeta+\mathrm{i})^{2m/\pi}\,\mathrm{e}^{-\mathrm{i}m} \tag{2.14}$$

as $\zeta \to -\mathrm{i}$, and similarly as $\zeta \to \mathrm{i}$. Thus altogether we write

$$\begin{aligned} z = \frac{\mathrm{i}\chi}{c} - H(\zeta-1) - \zeta(1-\zeta^2)\left(\frac{1}{\pi}+\tfrac{1}{2}H\right) \\ +(1-\zeta^2)^{2/3}[C''\zeta(1+\zeta^2)^{2m/\pi} + B'\zeta(1-\zeta^2)^{-1-\lambda} \\ +(a_0\zeta + a_1\zeta^3 + a_2\zeta^5 + \dots)]. \end{aligned} \tag{2.15}$$

Symmetry about a vertical line through the wave crest crest requires that

$$z(\zeta^*) = z^*(\zeta), \tag{2.16}$$

while the absence of even powers of ζ in the power series ensures that

$$z(-\zeta^*) = 2(H+1) - z^*(\zeta), \tag{2.17}$$

which is the condition for symmetry about the bottom $x = H+1$. The constant B' is related to the constant B of equation (2.9) by

$$B' = B\left(\frac{\pi}{c}\right)^{1/3+\lambda}. \tag{2.18}$$

We can now derive from equation (2.15) an expression for $\mathrm{d}z/\mathrm{d}\chi$ and substitute in the Bernoulli equation for constant pressure at the free surface, which may be written

$$(z+z^*)\frac{\mathrm{d}z}{\mathrm{d}\chi}\left(\frac{\mathrm{d}z}{\mathrm{d}\chi}\right)^* = 1 \tag{2.19}$$

to be satisfied on $|\zeta| = 1$, $\mathrm{Re}(\zeta) > 0$. Its reflected form is to be satisfied on $|\zeta| = 1$, $\mathrm{Re}(\zeta) < 0$, but because of the complete symmetry of the problem about $\mathrm{Re}(\zeta) = 0$ it is sufficient to require only that (2.19) be satisfied on the appropriate half-circumference; the reflected condition will then be satisfied automatically. So having introduced the reflected wave for the purpose of constructing the power series in (2.15) we now progress to a representation in which the physical free surface corresponds to the whole circumference of the unit circle by writing

$$\omega = \zeta^2. \tag{2.20}$$

We can express $\zeta = \omega^{1/2}$ as a Fourier series in $\arg(\omega)$ on the free surface $|\omega| = 1$ where we define

$$-\tfrac{1}{2}\pi < \arg(\omega^{1/2}) < \tfrac{1}{2}\pi. \tag{2.21}$$

Then we have

$$2\,\mathrm{Re}\,(z) = (1-\omega)^{2/3}\mathscr{A}(\omega)+\text{c.c.}, \tag{2.22}$$

where

$$\mathscr{A}(\omega) = H(1-\omega)^{1/3}(1+\omega^{1/2})^{-1}-\omega^{1/2}(1-\omega)^{1/3}\left(\frac{1}{\pi}+\tfrac{1}{2}H\right)$$
$$+\omega^{1/2}[B'(1-\omega)^{-1-\lambda}+C''(1+\omega)^{2m/\pi}+(a_0+a_1\omega+\ldots)] \tag{2.23}$$

and on differentiating (2.15) with the use of the relations (2.2) and (2.3) we obtain

$$-\mathrm{i}c\frac{\mathrm{d}z}{\mathrm{d}\chi} = (1-\omega)^{-1/3}\mathscr{B}(\omega), \tag{2.24}$$

where

$$\mathscr{B}(\omega) = (1-\omega)^{1/3}+\Bigg[(1-\omega)^{1/3}\left\{H+(1-3\omega)\left(\frac{1}{\pi}+\tfrac{1}{2}H\right)\right\}$$
$$-(1-\tfrac{7}{3}\omega)\{B'(1-\omega)^{-1-\lambda}+C''(1+\omega)^{2m/\pi}+(a_0+a_1\omega+\ldots)\}$$
$$-2\omega(1-\omega)\Bigg\{(\lambda+1)\,B'(1-\omega)^{-2-\lambda}+\frac{2m}{\pi}C''(1+\omega)^{2m/\pi-1}$$
$$+(a_1+2a_2\omega^2+3a_3\omega^3+\ldots)\Bigg\}\Bigg]\frac{\pi}{4}(1+\omega). \tag{2.25}$$

Substitution in equation (2.19) gives

$$(-\omega)^{1/3}\mathscr{A}(\omega)\,\mathscr{B}(\omega)\,\mathscr{B}(\omega^{-1})+\text{c.c.} = c^2 \tag{2.26}$$

on $|\omega| = 1$, $|\arg(-\omega)^{1/3}| < \pi/3$.

On truncating the power series in ω after $(N-5)$ terms we then have N unknowns:

$$c^2, H, m, B', C'', a_0, a_1 \ldots, a_{N-6} \tag{2.27}$$

to determine. We therefore require N equations relating the unknowns. We may take these to be (i)

$$H = \tfrac{1}{2}c^2 \tag{2.28}$$

which ensures that the free-surface condition is satisfied far from the crest; (ii) equation (2.13); (iii) a condition to ensure that the power series $(a_0+a_1\omega+--)$ is $o(1-\omega)^{-\lambda-1}$ as $\omega\to 1$, so that all the first-order asymptotic behaviour near the crest is included in the term $B'\zeta(1-\zeta^2)^{-\lambda-1}$ in equation (2.15). This condition (see Appendix A) is

$$\tfrac{5}{6}[2^{2m/\pi}C''+(a_0+a_1+\ldots)]+\frac{m}{\pi}2^{2m/\pi}C''+(a_1+2a_2+\ldots) = 0; \tag{2.29}$$

(iv) a similar condition on the asymptotic behaviour far from the crest, that is

$$\left(\frac{2}{\pi}+\tfrac{3}{2}H\right)-\frac{5}{3}2^{-1/3}[2^{-1-\lambda}B'+(a_0-a_1+a_2-\ldots)]$$
$$+2^{2/3}[(\lambda+1)\,2^{-2-\lambda}B'+(a_1-2a_2+3a_2-\ldots)] = 0; \tag{2.30}$$

(v) the $(N-4)$ equations obtained by equating the coefficients of the highest $(N-4)$ powers of ω in (2.26).

The system of nonlinear equations was solved as described in LHF1 for the highest deep-water wave. The results for c^2, together with the proportional error obtained

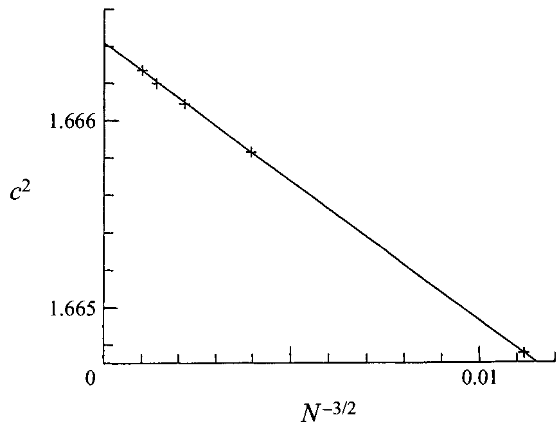

FIGURE 2. The highest solitary wave – plot of c^2 against $N^{-3/2}$.

		Percentage error in free-surface condition	
N	c^2	Maximum	Average
20	1.66475	4.0	0.08
20	1.66583	1.8	0.03
60	1.66609	1.3	0.02
80	1.66620	1.1	0.15
100	1.66627	0.9	0.01

TABLE 1. Values of c^2, for different values of N, and the proportional error in the free-surface condition, for the limiting solitary wave

n	a_n	n	a_n	n	a_n
0	0.6453	7	−0.0020	14	−0.0001
1	−0.1813	8	−0.0007	15	−0.0003
2	−0.0125	9	−0.0011	16	0.0001
3	−0.0154	10	−0.0004	17	−0.0002
4	−0.0034	11	−0.0006	18	0.0000
5	−0.0046	12	−0.0002	19	−0.0001
6	−0.0015	13	−0.0004		

$B' = 0.1147$, $C' = 0.1929$, $m = 1.0525$

TABLE 2. Values of B', C' and m and of the coefficients a_0 to a_{19} for the case $N = 60$

when the computed values of z and $dz/d\chi$ were substituted in the free-surface condition, are shown in table 1. Although convergence is slower than in the deep-water case, nevertheless c^2 appears to be converging to the value 1.666. Figure 2, in which c^2 is plotted against $N^{-3/2}$, suggests that the values can be extrapolated linearly to give $c^2 = 1.6664$. This agrees to four decimal places with later result $c^2 = 1.66639(4)$ given by Williams (1981) which is believed to be accurate to at least five decimal places.

Table 2 shows the values obtained for the first 20 coefficients a_i, as well as B', C' and m for the case $N = 60$.

	Fox (1977)	Williams (1981)
Phase-speed c	1.2909	1.29089
Wave height H	0.8332	0.83320
Mass M	1.968	1.97032
Impulse I	2.540	2.54346
Potential energy V	0.437	0.43767
Kinetic energy T	0.534	0.53501
Total energy E	0.971	0.97268

TABLE 3. Integral properties of the highest solitary wave as computed in §2, and as given by Williams (1981)

The numerical values of c and other integral quantities associated with the highest wave are shown in table 3. The method of derivation is given below in §5. Comparison with the definitive results of Williams (1981) shows that apart from the wave height H which is directly related to c^2, the other quantities were accurate to about three decimal places.

3. The almost-highest wave: matching procedure

The theory for the almost-highest solitary wave proceeds on precisely similar lines to the theory for deep-water waves, as given in LHF2. Thus we define a small parameter ϵ by

$$\epsilon^2 = q^2/(2c_l^2), \tag{3.1}$$

where q denotes the particle speed at the crest of the wave in the reference frame moving with the phase speed c, and c_l denotes the linear phase speed, in this case

$$c_l^2 = gD = 1, \tag{3.2}$$

with our choice of units. The flow is divided into three zones, as in figure 3: an inner zone I of radius $O(\epsilon^2)$ centred on the wave crest, in which typical lengths are of order ϵ^2 and typical velocities are $O(\epsilon)$; an intermediate zone II where typical lengths and velocities are of order ϵ and $\epsilon^{1/2}$ respectively; and an outer zone III where lengths and velocities are of order unity, in our system. Thus in zone I we have

$$z = \epsilon^2 z_I, \quad \chi = \epsilon\chi_I, \tag{3.3}$$

where z_1 and χ_1 are of order unity. The solution $z_1(\chi_1)$ is the lowest-order inner solution for the almost-highest wave, which was calculated in LHF1. From that paper we know that, for large values of χ_1, z_1 has the form

$$z_1 = (\tfrac{3}{2}\mathrm{i}\chi)^{2/3}(1+R), \tag{3.4}$$

where
$$R = C(\mathrm{i}\chi_I)^{-(1+\mathrm{i}\mu)} + C^*(-\mathrm{i}\chi_I^*)^{-(1-\mathrm{i}\mu)}, \tag{3.5}$$

C is a constant and μ is the positive root of

$$\frac{\pi\mu}{2}\tanh\frac{\pi\mu}{2} = \frac{\pi}{2\sqrt{3}}, \tag{3.6}$$

that is $\mu = 0.7143\ldots$.

On the other hand we have seen that in the outer zone III the highest wave ($\epsilon = 0$) has the form

$$z \propto (\tfrac{3}{2}\mathrm{i}\chi)^{2/3}[1+\gamma(\mathrm{i}\chi)^{-\lambda-1}], \tag{3.7}$$

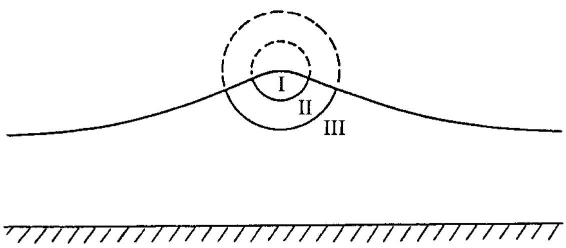

FIGURE 3. The almost-highest solitary wave – sketch showing regions of validity for the inner solution (zone 1), the outer solution (zone III) and for matching the two solutions (zone II).

where γ is a constant. We now assume that in the matching zone II both the inner asymptotic expansion, of which (3.4) is the lowest-order term, and the outer asymptotic expansion, of which (3.7) is the lowest term, are valid. So the appropriate asymptotic form in zone II as $\epsilon \to 0$ must contain all the terms of (3.4) and (3.7). That is, if we write

$$z = \epsilon z_{II}, \quad \chi = \epsilon^{3/2}\chi_{II}, \tag{3.8}$$

then

$$z_{II} \sim (\tfrac{3}{2}\mathrm{i}\chi_{II})^{2/3} + B\epsilon^{-3(1-\lambda)/2}(\mathrm{i}\chi_{II})^{-1/3-\lambda} + [A\epsilon^{3(1+\mathrm{i}\mu)/2}(\mathrm{i}\chi_{II})^{-1/3-\mathrm{i}\mu} + \text{c.c.}]. \tag{3.9}$$

Hence to the lowest-order outer solution we must add a correction term corresponding to the last expression in (3.9). In terms of the appropriate variables z and χ this will have the asymptotic form

$$z' \sim A\epsilon^{3(1+\mathrm{i}\mu)}(\mathrm{i}\chi)^{-1/3-\mathrm{i}\mu} + \text{c.c.} \tag{3.10}$$

Similarly a correction z'_I must be made to the inner flow z_r having the form

$$z'_I \sim B\epsilon^{-3(1+\lambda)}(\mathrm{i}\chi_I)^{-1/3-\lambda}. \tag{3.11}$$

4. The correction to the outer flow

The lowest-order corrections to the phase speed, momentum and other integral quantities of the solitary wave will come from the lowest-order corrections to the outer flow. Hence for small but positive ϵ we write in general

$$z = z_0(\mathrm{i}\chi) + [\epsilon^{3(1+\mathrm{i}\mu)}(1-\zeta^2)^{-1/3-\mathrm{i}\mu}(d_0\zeta + d_1\zeta^3 + \ldots) + \text{c.c.}], \tag{4.1}$$

where $z_0(\mathrm{i}\chi)$ is the solution given in §2 and the coefficients d_n are complex in general. We must also take

$$\left.\begin{aligned} c^2 &= c_0^2 + [\epsilon^{3(1+\mathrm{i}\mu)}c_1^2 + \text{c.c.}], \\ (H+\epsilon^2) &= H_0 + [\epsilon^{3(1+\mathrm{i}\mu)}H_1 + \text{c.c.}], \\ m &= m_0 + [\epsilon^{3(1+\mathrm{i}\mu)}m_1 + \text{c.c.}], \\ C'' &= C''_0 + [\epsilon^{3(1+\mathrm{i}\mu)}C''_1 + \text{c.c.}]. \end{aligned}\right\} \tag{4.2}$$

Lastly B' is defined to be equal to its lowest-order value B'_0, any change in the asymptotic behaviour as $\zeta \to \pm 1$ being absorbed in the correction power series in (4.1).

Then, modifying (2.22) we have

$$\begin{aligned} 2\,\mathrm{Re}(z) = (1-\omega)^{2/3}\mathscr{A}(\omega) + \epsilon^{3(1+\mathrm{i}\mu)}(1-\omega)^{-1/3}\mathscr{C}(\omega) \\ + \epsilon^{3(1-\mathrm{i}\mu)}(1-\omega)^{-1/3}\mathscr{C}^*(\omega) + \text{c.c.}, \end{aligned} \tag{4.3}$$

where $\mathscr{A}(\omega)$ is the lowest-order expression defined in §2,

$$\begin{aligned} \mathscr{C}(\omega) = (1-\omega)^{-\mathrm{i}\mu}\omega^{1/2}(d_0 + d_1\omega + \ldots) - H_1[(\omega^{1/2}-1) + \tfrac{1}{2}\omega^{1/2}(1-\omega)](1-\omega)^{1/3} \\ + \omega^{1/2}(1-\omega)(1+\omega)^{2m_0/\pi}C''_1 + \frac{2}{\pi}\omega^{1/2}(1-\omega)(1+\omega)^{2m_0/\pi}\ln(1+\omega)\,C''_0 m_1, \end{aligned} \tag{4.4}$$

n	d_n	n	d_n
0	$0.601-0.421\mathrm{i}$	10	$0.000-0.009\mathrm{i}$
1	$-0.475+0.442\mathrm{i}$	11	$0.000-0.008\mathrm{i}$
2	$0.033+0.059\mathrm{i}$	12	$-0.002-0.006\mathrm{i}$
3	$0.075+0.045\mathrm{i}$	13	$-0.002-0.006\mathrm{i}$
4	$0.034+0.006\mathrm{i}$	14	$-0.003-0.005\mathrm{i}$
5	$0.033-0.007\mathrm{i}$	15	$-0.003-0.004\mathrm{i}$
6	$0.015-0.010\mathrm{i}$	16	$-0.003-0.003\mathrm{i}$
7	$0.013-0.012\mathrm{i}$	17	$-0.003-0.003\mathrm{i}$
8	$0.005-0.011\mathrm{i}$	18	$-0.002-0.002\mathrm{i}$
9	$0.004-0.010\mathrm{i}$	19	$-0.003-0.002\mathrm{i}$

$$c_1^2 = -0.259-1.072\mathrm{i}, \qquad m_1 = -0.113-0.468\mathrm{i}$$
$$H_1 = -0.130-0.536\mathrm{i}, \qquad c_1' = -0.190+0.123\mathrm{i}$$

TABLE 4. Values of c_1^2, H_2, C_2' and m_1 and of the coefficients d_0 to d_{54} in the series for the first correction to the solitary wave, for the case $N = 60$

and
$$\mathscr{C}^*(\omega) = [\mathscr{C}(\omega^*)]^*. \tag{4.5}$$
Differentiating, we obtain
$$-\mathrm{i}c\frac{\mathrm{d}z}{\mathrm{d}\chi} = (1-\omega)^{-1/3}\mathscr{B}(\omega)+\epsilon^{3(1+\mathrm{i}\mu)}(1-\omega)^{-4/3}\mathscr{D}(\omega)-\epsilon^{3(1-\mathrm{i}\mu)}(1-\omega)^{-4/3}\mathscr{D}^*(\omega), \tag{4.6}$$
where $\mathscr{B}(\omega)$ is defined in §2 and $\mathscr{D}(\omega)$ is the expression given in Appendix B. We now substitute (4.3) and (4.6) in the free-surface condition (2.19) and collect terms involving $\epsilon^{3(1+\mathrm{i}\mu)}$. These give
$$\mathscr{F}(\omega)+\mathscr{F}(\omega^{-1}) = c_1^2(1-\omega)(-\omega)^{1/2}, \tag{4.7}$$
on $|\omega| = 1$, where
$$\mathscr{F}(\omega) = (-\omega)^{5/6}\mathscr{A}(\omega)\mathscr{B}(\omega)\mathscr{D}(\omega^{-1}) + (-\omega)^{1/6}\mathscr{A}(\omega)\mathscr{B}(\omega^{-1})\mathscr{D}(\omega)-(-\omega)^{-1/6}\mathscr{C}(\omega)\mathscr{B}(\omega)\mathscr{B}(\omega^{-1}). \tag{4.8}$$
The N complex unknowns
$$c_1^2, H_1, m_1, C_1'', d_0, d_1, \dots, d_{N-5}, \tag{4.9}$$
are then determined by N complex linear equations as follows. From the first-order correction to (2.28) and (2.13) we have
$$H_1 = \tfrac{1}{2}c_1^2 \tag{4.10}$$
and
$$m_0 c_1^2 = (\sec^2 m_0 - c_0^2)\, m_1. \tag{4.11}$$
From the matching condition (3.9) we have
$$d_0+d_1+d_2+\dots+d_{N-5} = A(c_0/\pi)^{-1/3-\mathrm{i}\mu} \tag{4.12}$$
and corresponding to equation (2.30) we have
$$2^{-1/3-\mathrm{i}\mu}[-\tfrac{1}{2}+\tfrac{1}{2}(\tfrac{1}{3}+\mathrm{i}\mu)](d_0-d_1+d_2-\dots)+2^{-1/3-\mathrm{i}\mu}(d_1-2d_1+3d_2-\dots)+\tfrac{3}{2}H_1 = 0 \tag{4.13}$$
Lastly we equate the first $(N-4)$ Fourier coefficients on each side of (4.7).

The above equations were solved for the case $N = 60$ using the values of
$$c_0^2, H_0, m_0 B', C_0'', a_0, \dots, a_{54} \tag{4.14}$$
determined in §2 for $N = 60$. Table 4 shows the values of the unknowns as obtained.

5. Formulae for integral quantities

From our numerical results, the first of equations (4.2) becomes, in real terms,

$$c^2 = c_0^2 + 2.2\epsilon^3 \cos(2.143 \ln \epsilon - 1.8), \tag{5.1}$$

the remainder being of order ϵ^4. It follows immediately from equation (3.1) and Bernoulli's equation

$$H = \tfrac{1}{2}(c^2 - q^2) \tag{5.2}$$

that

$$H + \epsilon^2 = H_0 + 1.1\epsilon^3 \cos(2.143 \ln \epsilon - 1.8). \tag{5.3}$$

The excess mass M is defined by

$$M \equiv \int_{-\infty}^{\infty} (H + \epsilon^2 - x)\, \mathrm{d}y = M_0 + M_1 \tag{5.4}$$

say, where x is measured from the Bernoulli zero-velocity level, not the crest, and

$$M_0 = \mathrm{Re} \int_0^{\infty} \mathrm{i}(H - \bar{z}_0) \frac{\mathrm{d}z_0}{\mathrm{d}\chi}\, \mathrm{d}\chi \tag{5.5}$$

and z_0 denotes the solution in §3. An overbar denotes the complex conjugate. For M_1 we have, to lowest order in ϵ,

$$M_1 = \mathrm{Re}\left\{ \epsilon^{3(1+\mathrm{i}\mu)} \int_0^{\infty} \left[\mathrm{i}(\bar{H}_1 - z_1^*) \frac{\mathrm{d}z_0}{\mathrm{d}\chi} + \overline{\mathrm{i}(H_0 - z_0)} \frac{\mathrm{d}z_1}{\mathrm{d}\chi} \right] \mathrm{d}\chi \right.$$
$$\left. + \epsilon^{3(1-\mathrm{i}\mu)} \int_0^{\infty} \left[\overline{\mathrm{i}(H_1 - z_1)} \frac{\mathrm{d}z_0}{\mathrm{d}\chi} + \overline{\mathrm{i}(H_0 - z_0)} \frac{\mathrm{d}z_1^*}{\mathrm{d}\chi} \right] \mathrm{d}\chi \right\}, \tag{5.6}$$

where

$$z_1 = (1 - \zeta^2)^{-1/3 - \mathrm{i}\mu} (d_0 \zeta + d_1 \zeta^3 + \ldots) \tag{5.7}$$

(see equation (4.1)) and z_1^* denotes the complex conjugate function of $\mathrm{i}\chi$ (or ω), i.e.

$$z_1^*(\mathrm{i}\chi) = \overline{z_1(\overline{\mathrm{i}\chi})}. \tag{5.8}$$

After evaluating the integrals in (5.6) we obtain finally (after correcting a sign in Fox 1977)

$$M = M_0 + 4.0\,\epsilon^3 \cos(2.143 \ln \epsilon - 2.6). \tag{5.9}$$

The impulse, or excess momentum, of the solitary wave is now easily obtained from Starr's (1947) relation

$$I = cM. \tag{5.10}$$

Thus on multiplying the right-hand sides of (5.1) and (5.9) and neglecting terms of order ϵ^4 or higher we have

$$I = I_0 + 6.4\,\epsilon^3 \cos(2.143 \ln \epsilon - 2.4). \tag{5.11}$$

Likewise the potential energy V can be found from the relation

$$3V = (c^2 - 1)\, M \tag{5.12}$$

(see Starr 1947; Longuet-Higgins 1974), to give

$$V = V_0 + 2.1\,\epsilon^3 \cos(2.143 \ln \epsilon - 2.1). \tag{5.13}$$

The kinetic energy T can most conveniently be calculated from

$$2T = c(I-\Gamma), \tag{5.14}$$

where Γ is the 'circulation' (see McCowan 1891; Longuet-Higgins 1974) defined by

$$\Gamma = [\phi - cy]_{-\infty}^{\infty}. \tag{5.15}$$

This yields

$$T = T_0 + 2.2\,\epsilon^3 \cos(2.143 \ln \epsilon - 2.0). \tag{5.16}$$

Lastly the total energy

$$E = T + V \tag{5.17}$$

is given by

$$E = E_0 + 4.3\,\epsilon^3 \cos(2.143 \ln \epsilon - 2.0). \tag{5.18}$$

However, because of the slow convergence of the coefficients d_n in table 4, the last decimal place in each of the above amplitude and phase constants must be considered as uncertain.

6. Discussion

Let us denote by Δc, ΔE, etc. the coefficients of ϵ^3 in the asymptotic expansions of c, E, etc. found in §5. For example if

$$c = c_0 + A\epsilon^3 \cos(2.143 \ln \epsilon - B) + O(\epsilon^4), \tag{6.1}$$

we write

$$A\,\mathrm{e}^{\mathrm{i}B} = \Delta c \tag{6.2}$$

and similarly for the other quantities, then it is immediately apparent that

$$\Delta c^2 = 2c_0 \Delta c \tag{6.3}$$

and

$$\Delta H = c_0 \Delta c \tag{6.4}$$

precisely. Also by differentiation of equations (5.10) and (5.12) we obtain the exact relations

$$\Delta I = M_0 \Delta c + c_0 \Delta M \tag{6.5}$$

and

$$3\Delta V = 2M_0 c_0 \Delta c + (c_0^2 - 1)\Delta M. \tag{6.6}$$

A further condition, not yet used, is derived from the differential relation

$$\mathrm{d}L = T\,\mathrm{d}c^2/c^2, \tag{6.7}$$

where L denotes the Lagrangian $(T-V)$; see Longuet-Higgins (1974) equation (D''). From this we obtain

$$\Delta L = (2T_0/c_0)\Delta c \tag{6.8}$$

with

$$\Delta L = \Delta T - \Delta V. \tag{6.9}$$

Lastly

$$\Delta E = \Delta T + \Delta V, \tag{6.10}$$

where E is the total energy. Thus we have seven relations, (6.3)–(6.6) and (6.8)–(6.10) connecting the nine complex quantities Δc, Δc^2, ΔH; ΔM, ΔI; ΔV, ΔT, ΔL and ΔE. For the present we leave out of account the relation for $\Delta\Gamma$ derived from equation (5.14).

From equations (6.8) and (6.9) and the calculated values of T_0 and c_0 from table 3 it becomes apparent that the numerical values for ΔT and ΔV quoted in §5 cannot be valid to more than one significant figure. However, an opportunity to improve the accuracy of all the constants came with the presentation by Tanaka (1995; Longuet-Higgins & Tanaka 1996) of a table of values for c, H, I and E calculated by an

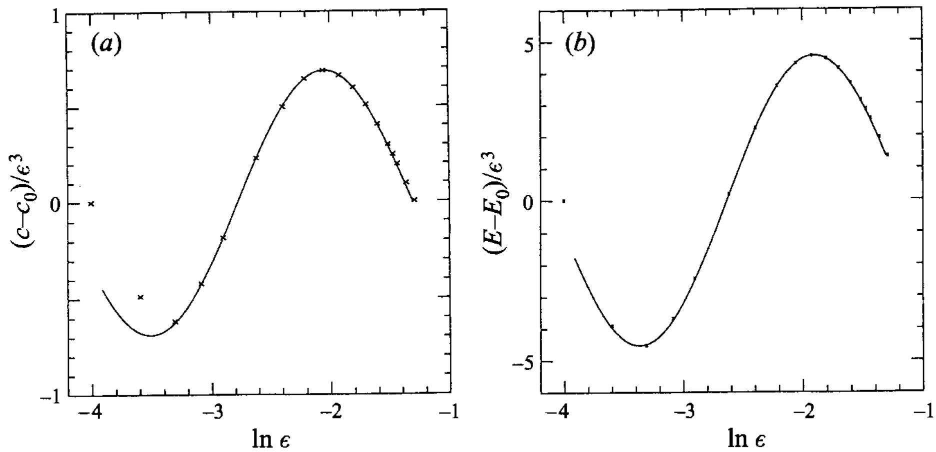

FIGURE 4. (a) Plot of $(c-c_0)/\epsilon^3$ versus ln ϵ. (b) plot of $(E-E_0)/\epsilon^2$ versus ln ϵ; ×, Data from Tanaka (1995). The sine-curve is fitted to the data.

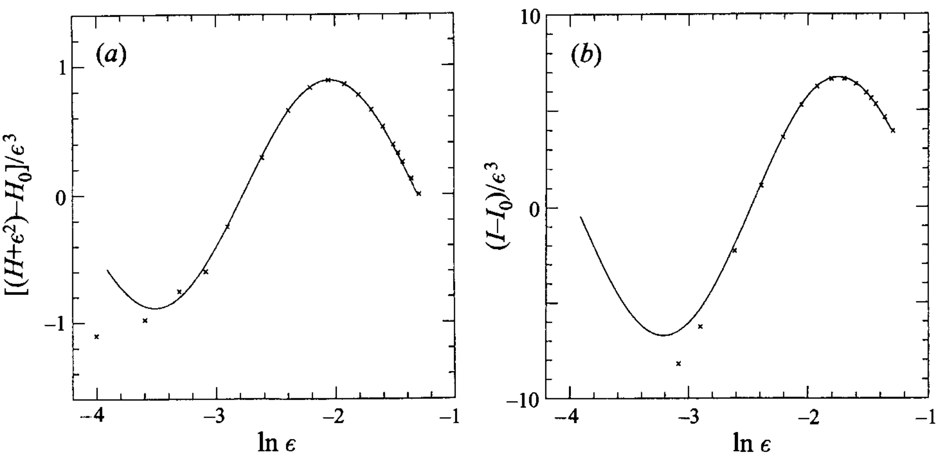

FIGURE 5. (a) Plot of $(H+\epsilon^2-H_0)/\epsilon^3$ versus ln ϵ; × denotes d_a from Tanaka (1995). The sine-curve is from equation (6.3). (b) Plot of $(I-I_0)/\epsilon^3$ versus ln ϵ; × data from Tanaka (1995). The sine-curve is from equation (6.5), etc.

independent method – essentially a refinement of the integral equation technique used in Tanaka (1986). In figure 4(a) we plotted the quantity $(c-c_0)/\epsilon^3$ against $\ln \epsilon$, where c is taken from Tanaka's data, c_0 has the value 1.29089 given by Williams (see table 3 above) and the abscissa is $\ln \epsilon$. According §5 we would expect this to yield a sine-wave of radian frequency 2.143. In figure 4 we have fitted the curve

$$f_c = 0.688 \cos(\theta - 1.90), \tag{6.11}$$

where

$$\theta = 2.143 \ln \epsilon. \tag{6.12}$$

Apart from the smallest values of ϵ which are affected by rounding errors, the fit is excellent, confirming the general theory of §§3 and 4 of the present paper. Figure 4(b) is a similar display for the total energy E, the data being fitted by the sine-curve

$$f_E = 4.530 \cos(\theta - 2.21). \tag{6.13}$$

In order to compute the remaining amplitude and phase constants and to check the accuracy of Tanaka's data we proceeded as follows. ΔH and Δc^2 were first derived from equations (6.3) and (6.4). Then equation (6.8) gave ΔL. The combination

$$\Delta E - \Delta L = 2\Delta V \tag{6.14}$$

of equations (6.9) and (6.10) gave ΔV and

$$\Delta E + \Delta L = 2\Delta T \tag{6.15}$$

gave ΔT. Then ΔM was found from (6.6) and ΔI from (6.7). This sequence was adopted in order to minimize the effect of errors in fitting a curve to ΔI, which has the largest amplitude. The resulting calculation gave the curves for y_H and y_I which are shown in figures 5(*a*) and 5(*b*) respectively. (These were not fitted to the corresponding data.)

The agreement seen in figures 5(*a*) and 5(*b*) is convincing and allows us to state with confidence the following set of numerical values:

$$\left.\begin{aligned} \Delta c &= 0.69\cos(\theta-1.90), \\ \Delta c^2 &= 1.78\cos(\theta-1.90), \\ \Delta H &= 0.89\cos(\theta-1.90); \end{aligned}\right\} \tag{6.16}$$

$$\left.\begin{aligned} \Delta M &= 4.44\cos(\theta-2.68), \\ \Delta I &= 6.77\cos(\theta-2.53); \end{aligned}\right\} \tag{6.17}$$

and

$$\left.\begin{aligned} \Delta V &= 2.00\cos(\theta-2.25), \\ \Delta T &= 2.54\cos(\theta-2.18), \\ \Delta E &= 4.53\cos(\theta-2.21), \\ \Delta L &= 0.57\cos(\theta-1.90). \end{aligned}\right\} \tag{6.18}$$

The nine vectors are shown in the complex plane in figure 6. For clarity we have omitted the prefix Δ. The origin is indicated by O and arrows indicate parallel lines. Note that IM is not parallel to ET.

Any two non-parallel vectors in figure 6 may be taken as base vectors and then all of the other vectors may be expressed in terms of them. For instance if Δc and ΔM are chosen as base vectors, then the situation is as shown in figure 7.

Finally we have evaluated the complete asymptotic expressions for c, H, M, I and E and have plotted them in figure 8(*a–e*). The abscissa in each diagram is chosen to be the parameter

$$\eta = 1 - q^2/gD = 1 - 2\epsilon^2 \tag{6.19}$$

introduced by Longuet-Higgins & Fenton (1994), where it is denoted by ω. This parameter has the advantage that it runs from 0 to 1 precisely throughout the range of solitary waves, $\eta = 0$ corresponding to zero amplitude and $\eta = 1$ to waves of limiting height. In figure 8(*a, b, d, e*) we have also plotted the values of C, H, I and E computed by Tanaka (1995). Finally in figures 9(*a*) and 9(*b*) we show enlarged plots of c and E close to the steepest wave.

As pointed out in §1, particular interest attaches to the turning points of the total energy E since, as shown by Tanaka (1986, 1995), the first and second turning points

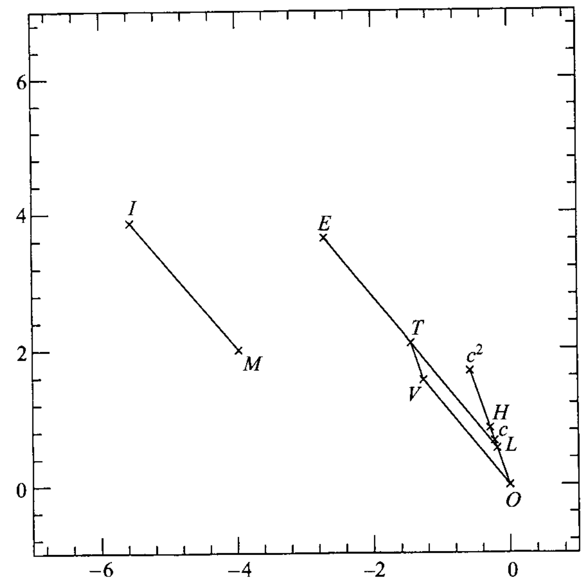

FIGURE 6. Relationship of the coefficients Δc, Δc^2, ΔH; ΔM, ΔI; ΔV, ΔT, ΔE and ΔL, in the complex plane. For clarity the prefixes Δ are omitted.

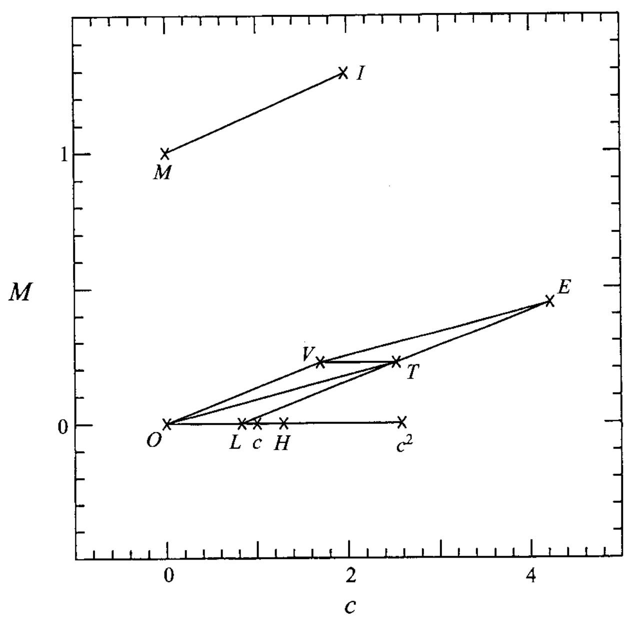

FIGURE 7. Similar to figure 6, but using Δc and ΔM as base vectors.

mark the onset of the first and second modes of instability of a solitary wave. A similar conclusion probably applies to the higher turning points. In table 5 we show the values of the first three turning points of the energy as predicted by equation (5.17), see Appendix C. The second and third turning points are given more accurately than the first.

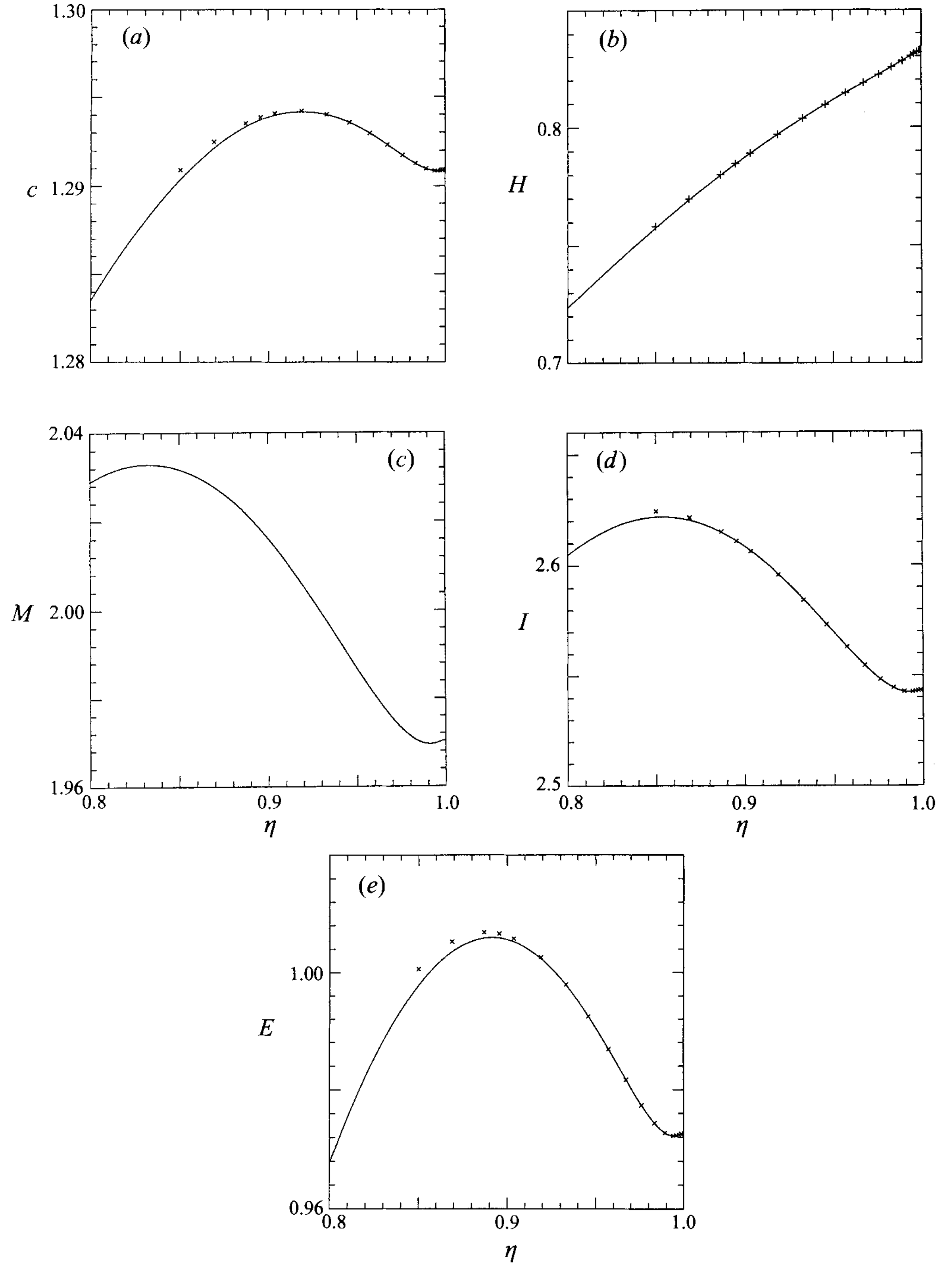

FIGURE 8. (*a*) The phase-speed *c*, (*b*) the wave height *H*, (*c*) the added mass *M*, (*d*) the impulse *I*, and (*e*) the total energy *E*, as functions of η for steep waves ($0.8 \leqslant \eta \leqslant 1$). Full curve corresponds to the asymptotic expression (5.1). Plotted points correspond to the integral-equation calculations by Tanaka (1995).

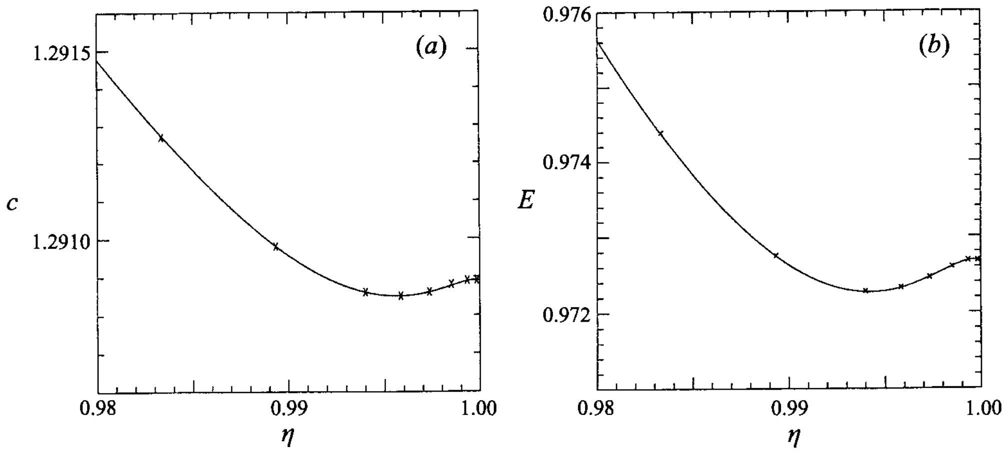

FIGURE 9. (*a*) Enlargement of figure 8(*a*), showing the first minimum of c. (*b*) Enlargement of figure 8(*e*), showing the first minimum of E.

	n	η	E	H
1	1st max	0.8916	1.0059	0.7824
2	1st min	0.99423	0.97227	0.83027
3	2nd max	0.99969	0.97269	0.83305

TABLE 5. Stationary values of the total energy E

7. Conclusions

We have shown that integral properties of steep solitary waves behave in an oscillatory manner as they approach the limiting configuration, just as was found previously by Longuet-Higgins & Fox (1977, 1978) for gravity waves in deep water. In fact the asymptotic theory of the 'almost-highest wave' is evidently applicable to steady progressive waves in water of arbitrary, uniform depth, though in any particular case the details must be worked out separately.

The constants in the asymptotic formulae for the phase speed, mass, momentum and energy of the solitary wave which were given by Fox (1977) were accurate to no more than one decimal place. However, knowing the form of the solution we have been able to test the accuracy of the recent numerical calculations by Tanaka (1995) and to use these calculations to evaluate the constants in the asymptotic formulae to at least two decimal places.

The verification of the accuracy has been assisted by the use of certain identical relations between the constants as pointed out in §6. Each of the quantities ΔC, ΔH, ΔI, etc. is represented by a two-dimensional vector, and the relations between them are such that when any two vectors are given the rest are linearly dependent on them.

The asymptotic formulae enable the turning points in each of the integral properties (as functions of the wave steepness) to be determined accurately. Of special interest are the turning points in the energy density E; the nth turning-point in E is the starting point for a new normal mode of instability of the crest of the wave, as was shown numerically by Tanaka (1986, 1995) in the cases $n = 1$ and 2 respectively. Indeed for the solitary wave these 'crest instabilities', for general n, appear to be the only types possible.

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Appendix A. Derivation of equation (2.29)

The last group of terms in equation (2.15) is designed to give the required form (2.9) as $\chi \to 0$, with all the $\chi^{-1/3-\lambda}$ behaviour included in the B' term. However the presence of a power series in ζ multiplying the leading-order $(1-\zeta^2)^{2/3}$ term in (2.15) will in general give rise to $\chi^{5/3}$ terms which are very close in behaviour to the B' term. To avoid this, we therefore impose a condition to suppress these terms by insisting that, if

$$(1-\zeta^2)^{2/3}[C''\zeta(1+\zeta^2)^{2m/\pi}+(a_0\zeta+a_1\zeta^3+a_3\zeta^5+\dots)] = \chi^{2/3}F(\chi) \tag{A 1}$$

then

$$\left(\frac{\mathrm{d}F}{\mathrm{d}\chi}\right)_{\chi=0} = 0. \tag{A 2}$$

Evaluation of this derivative requires an expression for the second-order behaviour of $(\zeta-1)$ with χ obtained from equation (2.2). This is

$$(\zeta-1) = -\frac{\mathrm{i}\pi\chi}{2c}-\frac{\pi^2\chi^2}{8c^2}. \tag{A 3}$$

Using this expression we find that

$$F(\chi) = \text{const.}\times\left(1-\frac{\mathrm{i}\pi\chi}{4c}\right)^{4/3}\zeta[C''(1+\zeta^2)^{2m/\pi}+(a_0+a_1\zeta^2+a_2\zeta^4+\dots)] \tag{A 4}$$

whose derivative at $\chi = 0$ gives us equation (2.29).

Appendix B. Definition of the function $\mathscr{D}(\omega)$ in equation (4.6)

$$\begin{aligned}\frac{4}{\pi(1+\omega)}\mathscr{D}(\omega) = {} & (1-\omega)^{-\mathrm{i}\mu}[(\mathrm{i}\mu+\tfrac{1}{3})\,2\omega(d_0+d_1\omega+d_2\omega^2+\dots) \\ & +(1-\omega)(d_0+d_1\omega+\dots)+2\omega(1-\omega)(d_1+2d_2 \\ & \omega+\dots)]-\tfrac{3}{2}H_1(1-\omega)^{7/3} \\ & +C_1''\left[\frac{2m_0}{\pi}2\omega(1-\omega)-\tfrac{4}{3}\omega(1+\omega)-(1-\omega^2)\right](1-\omega)(1+\omega)^{2m_0/\pi-1} \\ & +C_0''\frac{2m_0}{\pi}\left[\left\{(1-\omega^2)-\tfrac{4}{3}\omega(1+\omega)+\frac{4m_0}{\pi}\omega(1-\omega)\right\}\ln(1+\omega)\right. \\ & \left.-2\omega(1-\omega)\right](1-\omega)(1+\omega)^{2m_0/\pi-1}.\end{aligned} \tag{B 1}$$

Appendix C. Calculation of turning points

Any quantity of the form

$$f(\epsilon) \equiv f_0+A\epsilon^3\cos(\nu\ln\epsilon+B) \tag{C 1}$$

has a turning point when

$$\tan(\nu\ln\epsilon-B) = 3/\nu. \tag{C 2}$$

Hence

$$\nu \ln \epsilon = 0.95051 + B - n\pi, \tag{C 3}$$

where n is a positive integer, and

$$\epsilon = 1.558\ (1.594)^B (0.2308)^n. \tag{C 4}$$

The corresponding value of f is

$$f = f_0 + (-1)^{n-1}\, 0.5813 A\epsilon^3. \tag{C 5}$$

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